Some Possibility and Impossibility Results related to Discrete Fourier type transforms in Quantum Information

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Abstract

Very recently the most general ensemble of qubits are identified using the notion of linearity; any of these qubits gets accepted by a Hadamard gate to generate the equal superposition of the qubit and its orthogonal. Towards more generalization, we investigate the possibility and impossibility results related to Discrete Fourier Transform (DFT) type of operations for a more general set up of qutrits.

Keywords: Unitary operations, Fourier Transform, Qubits, Qutrits, Qudits, Hadamard Transformation.

1 Introduction

The two very important basic building blocks in the area of quantum computation and quantum information are qubits (in general qudits for higher dimensions) and the quantum gates. These basic building blocks of quantum computers are believed to be computationally stronger than their classical counterparts. One of the most important quantum gates in this paradigm is the Hadamard gate (that introduces equal superposition of the input state and its orthogonal) which has found wide applications in computer and communication sciences [8]. One may refer to a number of seminal papers in quantum computation and information theory where Hadamard transform has been used [5, 3, 6, 4]. Shor's polynomial time algorithm for factoring and discrete logarithm [11] is based on Fourier transform which is a generalization of the Hadamard transform in higher dimensions. One should also note that the Toffoli and Hadamard gates comprise the simplest quantum universal set of gates [10, 2]. Thus, the role played by the Hadamard gate (and more generally Fourier transform) in quantum information theory is indeed significant.

Pati [9] has observed that one can not design a universal Hadamard gate for an arbitrary unknown qubit as linear superposition of an arbitrary unknown state $|\psi\rangle$ with its orthogonal complement $|\psi_{\perp}\rangle$ is not achievable. However, it was noted that if one considers qubit states from the polar or equatorial great circles on a Bloch sphere, then it is possible to design Hadamard type of gates. By a Hadamard 'type' gate we mean a unitary matrix that is not exactly a Hadamard matrix, but it still creates an equal superposition (up to a sign or a phase) of a qubit and its complement to produce two orthogonal states. Later Song et. al. [12] have tried to implement the Hadamard gate in a probabilistic manner for any unknown state chosen from a set of linearly independent states. Further to Pati's work, very recently Maitra and Parashar [7] identified the most general class of qubit states, for which the Hadamard gate works in a deterministic fashion. Further, it is also shown in [7] that certain Hadamard 'type' transformations are indeed possible for arbitrary states when partial information is available.

Based on long standing interest in possibility and impossibility results in the field of quantum information, it is motivating to study these results in higher dimension. To be more precise one may study how the results of [9, 7] can be generalized for qudits when we consider the Fourier transformation. It has been clearly commented in [9] that one may try to extend the limitations and possible operations for higher dimensional quantum systems. We study these issues for qutrits in Section 2 and towards the end present a generalized result for qudits too.

2 Qutrit results

To study the possibility and impossibility results in higher dimensions, we mainly concentrate on qutrits in this section. The most general form of a qutrit is [1]

$$|\psi\rangle = \cos\gamma_1|0\rangle + \sin\gamma_1\cos\gamma_2\, e^{i\delta}|1\rangle + \sin\gamma_1\sin\gamma_2\, e^{i\phi}|2\rangle, \tag{1}$$

where $0 \le \gamma_1, \gamma_2 \le \frac{\pi}{2}$ and $0 \le \delta, \phi \le 2\pi$.

The Discrete Fourier Transform (DFT) can be seen as the Hadamard transform over higher dimensional space. It is defined as

$$|j\rangle = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{\frac{2\pi ijk}{n}}|k\rangle = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\Gamma^{jk}|k\rangle,$$

where $\Gamma = e^{\frac{2\pi i}{n}}$, n being the dimension of the Hilbert space.

For a qutrit this can be seen as follows when $\Gamma = e^{\frac{2\pi i}{3}}$.

$$\begin{aligned} U(|0\rangle) &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \\ U(|1\rangle) &= \frac{1}{\sqrt{3}}(|0\rangle + \Gamma|1\rangle + \Gamma^2|2\rangle), \\ U(|2\rangle) &= \frac{1}{\sqrt{3}}(|0\rangle + \Gamma^2|1\rangle + \Gamma|2\rangle). \end{aligned} \tag{2}$$

That is $U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \Gamma & \Gamma^2 \\ 1 & \Gamma^2 & \Gamma \end{bmatrix}$. The question is "is it possible to get a unitary transformation for a set of three arbitrary qutrits which are orthogonal", i.e., is it possible to get a unitary transformation U such that

$$\begin{aligned} U(|\psi_0\rangle) &= \frac{1}{\sqrt{3}}(|\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle), \\ U(|\psi_1\rangle) &= \frac{1}{\sqrt{3}}(|\psi_0\rangle + \Gamma|\psi_1\rangle + \Gamma^2|\psi_2\rangle), \\ U(|\psi_2\rangle) &= \frac{1}{\sqrt{3}}(|\psi_0\rangle + \Gamma^2|\psi_1\rangle + \Gamma|\psi_2\rangle). \end{aligned} \tag{3}$$

We will show that it is not true in general. Let us consider two sets of mutually orthogonal qutrits $\{v_k, v'_k, v''_k\}$ and $\{w_j, w'_j, w''_j\}$. If DFT is possible in general, then
$U(v_k) = \frac{1}{\sqrt{3}}(v_k + v'_k + v''_k)$,
$U(w_j) = \frac{1}{\sqrt{3}}(w_j + w'_j + w''_j)$,
$U(v'_k) = \frac{1}{\sqrt{3}}(v_k + \Gamma v'_k + \Gamma^2 v''_k)$,
$U(w'_j) = \frac{1}{\sqrt{3}}(w_j + \Gamma w'_j + \Gamma^2 w''_j)$,
$U(v''_k) = \frac{1}{\sqrt{3}}(v_k + \Gamma^2 v'_k + \Gamma v''_k)$,
$U(w''_j) = \frac{1}{\sqrt{3}}(w_j + \Gamma^2 w'_j + \Gamma w''_j)$,
and hence $\langle v_k|w_j\rangle = \langle v_k|U^\dagger U|w_j\rangle = \frac{1}{\sqrt{3}}(\langle v_k|w_j\rangle + \langle v_k|w'_j\rangle + \langle v_k|w''_j\rangle + \langle v'_k|w_j\rangle + \langle v'_k|w'_j\rangle + \langle v'_k|w''_j\rangle + \langle v''_k|w_j\rangle + \langle v''_k|w'_j\rangle + \langle v''_k|w''_j\rangle)$, which is not true in general.

As example, one can take
$v_k = \frac{1}{\sqrt{6}}(|0\rangle + 2e^{i\delta_k}|1\rangle + e^{i\phi_k}|2\rangle)$,
$w_j = \frac{1}{\sqrt{3}}(|0\rangle + e^{i\delta_j}|1\rangle + e^{i\phi_j}|2\rangle)$,
$v'_k = \frac{1}{\sqrt{11}}(|0\rangle + e^{i\delta_k}|1\rangle - 3e^{i\phi_k}|2\rangle)$,
$w'_j = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma e^{i\delta_j}|1\rangle + \Gamma^2 e^{i\phi_j}|2\rangle)$,
$v''_k = \frac{1}{\sqrt{66}}(7|0\rangle - 4e^{i\delta_k}|1\rangle + e^{i\phi_k}|2\rangle)$,
$w''_j = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma^2 e^{i\delta_j}|1\rangle + \Gamma e^{i\phi_j}|2\rangle)$,
and then check that $\langle v_k|w_j\rangle \neq \frac{1}{\sqrt{3}}(\langle v_k|w_j\rangle + \langle v_k|w'_j\rangle + \langle v_k|w''_j\rangle + \langle v'_k|w_j\rangle + \langle v'_k|w'_j\rangle + \langle v'_k|w''_j\rangle + \langle v''_k|w_j\rangle + \langle v''_k|w'_j\rangle + \langle v''_k|w''_j\rangle)$ in this case. Thus we have the following result.

Theorem 1 *It is not possible to get a generalized Discrete Fourier Transform for qutrits as given by U in Equation 3.*

After getting the impossibility result in general, we now consider the well known equatorial ensembles as a restricted case and try to obtain a "DFT like transformation" for this particular ensemble.

For the equatorial qutrits the inner product laws are:
$\langle v_k|w_j\rangle = \langle v'_k|w'_j\rangle = \langle v''_k|w''_j\rangle$,
$\langle v_k|w'_j\rangle = \langle v''_k|w_j\rangle = \langle v'_k|w''_j\rangle$, and

$\langle v_k|w''_j\rangle = \langle v'_k|w_j\rangle = \langle v''_k|w'_j\rangle$.

Let $U(|\psi_0\rangle) = \frac{1}{\sqrt{3}}(|\psi_0\rangle + \alpha|\psi_1\rangle + \beta|\psi_2\rangle)$. Now for two sets of mutually orthogonal equatorial qutrits $\{v_k, v'_k, v''_k\}$ and $\{w_j, w'_j, w''_j\}$ we get the following.

Let $U(v_k) = \frac{1}{\sqrt{3}}(v_k + \alpha v'_k + \beta v''_k)$ and $U(w_j) = \frac{1}{\sqrt{3}}(w_j + \alpha w'_j + \beta w''_j)$. Thus

$\langle v_k|w_j\rangle = \frac{1}{3}(\langle v_k|w_j\rangle + \alpha\langle v_k|w'_j\rangle + \beta\langle v_k|w''_j\rangle + \alpha^*\langle v'_k|w_j\rangle + \alpha\alpha^*\langle v'_k|w'_j\rangle + \beta\alpha^*\langle v'_k|w''_j\rangle + \beta^*\langle v''_k|w_j\rangle + \alpha\beta^*\langle v''_k|w'_j\rangle + \beta\beta^*\langle v''_k|w''_j\rangle)$. Further $\langle v_k|w_j\rangle = \frac{1}{3}((1+\alpha\alpha^*+\beta\beta^*)\langle v_k|w_j\rangle + (\alpha + \beta^* + \beta\alpha^*)\langle v_k|w'_j\rangle + (\beta + \alpha^* + \alpha\beta^*)\langle v_k|w''_j\rangle)$. For the left hand and right hand side to be equal,

$1 + \alpha\alpha^* + \beta\beta^* = 3$, i.e., $\alpha\alpha^* + \beta\beta^* = 2$,

$\alpha + \beta^* + \beta\alpha^* = 0$ and

$(\beta + \alpha^* + \alpha\beta^*) = 0$.

Note that $\alpha = \beta = \Gamma = e^{\frac{2\pi i}{3}}$ satisfy the above three equations.

Thus $U(|\psi_0\rangle) = \frac{1}{\sqrt{3}}(|\psi_0\rangle + \Gamma|\psi_1\rangle + \Gamma|\psi_2\rangle)$.

Consider the equatorial qutrits

$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + e^{i\delta}|1\rangle + e^{i\phi}|2\rangle)$,

$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma e^{i\delta}|1\rangle + \Gamma^2 e^{i\phi}|2\rangle)$, and

$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma^2 e^{i\delta}|1\rangle + \Gamma e^{i\phi}|2\rangle)$.

Now consider the unitary transformation $U = \frac{1}{\sqrt{3}}\begin{bmatrix} 2\Gamma+1 & 0 & 0 \\ 0 & 2+\Gamma^2 & 0 \\ 0 & 0 & 2+\Gamma^2 \end{bmatrix}$.

One can check that,

$U(|\psi_0\rangle) = \frac{1}{\sqrt{3}}(|\psi_0\rangle + \Gamma|\psi_1\rangle + \Gamma|\psi_2\rangle)$,

$U(|\psi_1\rangle) = \frac{1}{\sqrt{3}}(|\psi_1\rangle + \Gamma|\psi_0\rangle + \Gamma|\psi_2\rangle)$ and

$U(|\psi_2\rangle) = \frac{1}{\sqrt{3}}(|\psi_2\rangle + \Gamma|\psi_0\rangle + \Gamma|\psi_1\rangle)$.

Renaming, $U(|\psi_0\rangle)$, $U(|\psi_1\rangle)$ and $U(|\psi_2\rangle)$ as $|\psi_A\rangle$, $|\psi_B\rangle$ and $|\psi_C\rangle$, one can check that $\langle\Psi_A|\Psi_B\rangle = \langle\Psi_A|\Psi_C\rangle = \langle\Psi_B|\Psi_C\rangle = 0$, i.e., they are orthogonal to each other.

If we consider the other solution $\alpha = \beta = \Gamma^2 = e^{\frac{4\pi i}{3}}$, then the form of the matrix is $U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1+2\Gamma^2 & 0 & 0 \\ 0 & 2+\Gamma & 0 \\ 0 & 0 & 2+\Gamma \end{bmatrix}$.

Thus we get a possibility result for equatorial qutrits which provides a DFT kind of transformation. In case of equatorial qubits, the transformation is $U|\psi\rangle \rightarrow |\psi\rangle + \Gamma^{\frac{1}{2}}|\overline{\psi}\rangle$, where $\Gamma = e^{\frac{2\pi i}{2}} = e^{\pi i}$, where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+\Gamma^{\frac{1}{2}} & 0 \\ 0 & 1-\Gamma^{\frac{1}{2}} \end{bmatrix}$. This has been referred in [9]. Here $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ and $|\overline{\psi}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \Gamma e^{i\phi}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\phi}|1\rangle)$.

Thus our result clearly extends the work of [9] for higher dimension (qutrits).

2.1 Pairwise Hadamard Type Operation

We like to extend the idea for qutrits where an equatorial qutrit will be the input and the output will be the superposition of the qutrit and one of its orthogonals in rotational manner.

Let us consider the equatorial qutrits $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + e^{i\delta}|1\rangle + e^{i\phi}|2\rangle)$, $|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma e^{i\delta}|1\rangle + \Gamma^2 e^{i\phi}|2\rangle)$, and $|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \Gamma^2 e^{i\delta}|1\rangle + \Gamma e^{i\phi}|2\rangle)$.

Now consider the unitary transformation $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i\Gamma & 0 \\ 0 & 0 & 1+i\Gamma^2 \end{bmatrix}$.

One can check that

$U(|\psi_0\rangle) = \frac{1}{\sqrt{2}}(|\psi_0\rangle + i|\psi_1\rangle)$,

$U(|\psi_1\rangle) = \frac{1}{\sqrt{2}}(|\psi_1\rangle + i|\psi_2\rangle)$ and

$U(|\psi_2\rangle) = \frac{1}{\sqrt{2}}(|\psi_2\rangle + i|\psi_0\rangle)$.

Note that the states $U(|\psi_0\rangle), U(|\psi_1\rangle), U(|\psi_2\rangle)$ are not orthogonal to each other.

The similar technique can be extended for any dimensions, i.e., for the qudits too.

Note that if $\Gamma = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$, i.e., in the qubit case, U reduces to

$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$.

This means that the qubit matrix for equatorial great circle is embedded in the qutrit one. This implies that remote state preparation is also valid for equatorial qutrits but not for polar qutrits, as there is no such similarity in the polar case.

Towards the end we like to present a result for the most general set up of qudits. For qudits, the cyclic transformations are

$U(|\psi_0\rangle) = \frac{1}{\sqrt{2}}(|\psi_0\rangle + i|\psi_1\rangle)$,

$\ldots$,

$U(|\psi_r\rangle) = \frac{1}{\sqrt{2}}(|\psi_r\rangle + i|\psi_{r+1}\rangle)$,

$\ldots$,

$U(|\psi_{n-1}\rangle) = \frac{1}{\sqrt{2}}(|\psi_{n-1}\rangle + i|\psi_0\rangle)$,

where $|\psi_r\rangle = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\Gamma^{rk}|k\rangle$. So the form of U in this case will be

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+i & 0 & 0 & \ldots & 0 \\ 0 & 1+i\Gamma & 0 & \ldots & 0 \\ 0 & 0 & 1+i\Gamma^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1+i\Gamma^{n-1} \end{bmatrix}.$$

However, one should note that the states $U(|\psi_0\rangle), \ldots, U(|\psi_r\rangle), \ldots, U(|\psi_{n-1}\rangle)$ are non orthogonal.

3 Conclusion

In this paper we have studied the possibility and impossibility results related to Discrete Fourier Transform type operations. We study the possibility and impossibility results for DFT type of operations in the case of qutrits. Towards the end we present a result on qudits too.

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